

On the String Equation of the BKP Hierarchy

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Abstract

The Adler-Shiota-van Moerbeke formula is employed to derive the W -constraints for the p -reduced BKP hierarchy constrained by the string equation. We also provide the Grassmannian description of the string equation in terms of the spectral parameter.

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1 Introduction

String equation is an important constrained condition connecting integrable hierarchies with solvable string theories and intersection theory (see e.g. [8, 28] and references therein). The most famous example is the KdV hierarchy constrained by a string equation whose solutions corresponds to partition function of 2d quantum gravity or generating function of intersection number[29, 30, 15](see also [13]). Since the solutions of KdV hierarchy can be characterized by a single function called τ -function, it turns out that the constrained KdV hierarchy can be written as Virasoro constraints on the τ -function, in which the lowest one corresponds to the string equation[9, 11, 16, 17]. For higher order KdV hierarchy (i.e. p -reduced KP hierarchy with $p > 2$) the associated τ -function satisfies the W -constraints so that the τ -function is a null-vector of a set of W_p^+ -algebra which contains the Virasoro algebra as a subalgebra with central charge $c = p$ [10, 1, 28]. It was pointed out[16, 17, 5] that the string equation associated with the KP/KdV hierarchy can be traced back to additional symmetries (or non-isospectral flows)of the hierarchies proposed by Orlov and Schulman[18](see also [7]). Furthermore, Adler-Shiota-van Moerbeke [2, 3] and Dickey[6] showed that additional symmetries acting on wave function are connected with Sato's Bäcklund symmetries acting on tau function. These properties provide a useful tool to study the solutions to the string equation of the p -reduced KP hierarchy with $p > 2$ [1] that generalize Kontsevich's result for the KdV case.

In the present work, we like to study the string equation associated with the BKP hierarchy[4, 24]. The BKP hierarchy possesses many integrable structures as the KP hierarchy such as Lax formulation, tau function, Hirota bilinear equation, fermion representation, soliton and quasi-periodic solutions, etc. In [26] van de Leur provided the Adler-Shiota-van Moerbeke formula for the BKP hierarchy. He also studied the W -constraint for the p -reduced BKP hierarchy in terms of the twisted affine Lie algebra $\hat{sl}_n^{(2)}$ [27]. In this work, we follow closely the work by Adler-van Moerbeke[1] and Dickey[5] to reconsider the string equation of the BKP hierarchy from the point of view of additional symmetries in Lax-Orlov-Schulman formulation. In particular, we shall show that the invariance of the p -reduced BKP hierarchy with respect to a particular additional symmetry is crucial for obtaining the string equation which together with the Adler-Shiota-van Moerbeke formula implies the W -constraints of the τ function. In the end of the work we also give a Grassmannian description of the string equation in terms of the spectral parameter.

The paper is organized as follows. In section 2, we recall some basic notions for the BKP hierarchy such as Lax formulation, (adjoint-) wave function, τ function and (differential) Fay identity. In section 3, we introduce the Orlov-Schulman operator for additional symmetries

and vertex operators for Bäcklund symmetries of the BKP hierarchy. We give a simple expression of the generators of vertex operator using the Faà di Bruno polynomials and provide the Adler-Shiota-van Moerbeke formula that connects additional symmetries acting on wave function with those Sato's Bäcklund symmetries acting on tau function. In section 4, we show that the solution of the p -reduced BKP hierarchy constrained by the string equation can be characterized by a vacuum condition so that the associated tau function is annihilated by a set of differential operators. In section 5, We show that these differential operators constitute a W_p^{B+} algebra which contains a Virasoro algebra as a subalgebra with central charge $c = p$. In section 6, we provide the Grassmannian description of the string equation in terms of the spectral parameter. Section 7 is devoted to the concluding remarks.

2 BKP hierarchy

In this section we recall some basic properties for the BKP hierarchy [4]. We shall follow the notations used in the previous work[25]. The BKP hierarchy can be formulated in Lax form as

$$\partial_{2n+1}L = [B_{2n+1}, L], \quad B_{2n+1} = (L^{2n+1})_+, \quad n = 0, 1, 2, \dots \quad (1)$$

where the Lax operator is defined by

$$L = \partial + u_1\partial^{-1} + u_2\partial^{-2} + \dots, \quad (2)$$

with coefficient functions u_i depending on the time variables $t = (t_1 = x, t_3, t_5, \dots)$ and satisfies the constraint

$$L^* = -\partial L \partial^{-1}. \quad (3)$$

Here and the rest of the paper we will use the notations: $(\sum_i a_i \partial^i)_+ = \sum_{i \geq 0} a_i \partial^i$, $(\sum_i a_i \partial^i)_- = \sum_{i < 0} a_i \partial^i$, $(\sum_i a_i \partial^i)_{[k]} = a_k$, $\text{res}(\sum_i a_i \partial^i) = a_{-1}$ and $(\sum_i a_i \partial^i)^* = \sum_i (-\partial)^i a_i$. It can be shown [4] that the constraint (3) is equivalent to the condition $(B_{2n+1})_{[0]} = 0$.

The Lax equation (1) is equivalent to the compatibility condition of the linear system

$$Lw(t, z) = zw(t, z), \quad \partial_{2n+1}w(t, z) = B_{2n+1}w(t, z), \quad (4)$$

where $w(t, z)$ is called wave function (or Baker function) of the system and z is the spectral parameter. The whole hierarchy can be expressed in terms of a dressing operator, the so-called Sato's operator W , so that

$$L = W \partial W^{-1}, \quad W = 1 + \sum_{j=1} w_j \partial^{-j},$$

and the Lax equation is equivalent to the Sato's equation

$$\partial_{2n+1}W = -(L^{2n+1})_-W, \quad (5)$$

with constraint[23, 24]

$$W^*\partial W = \partial. \quad (6)$$

Let the solutions of the linear system (4) be the form

$$w(t, z) = We^{\xi(t, z)} = \hat{w}(t, z)e^{\xi(t, z)}, \quad (7)$$

where $\xi(t, z) = \sum_{i=0} t_{2i+1}z^{2i+1}$ and $\hat{w}(t, z) = 1 + w_1/z + w_2/z^2 + \dots$. Then $w(t, z)$ is a wave function of the BKP hierarchy if and only if it satisfies the bilinear identity[4]

$$\text{res}_z(z^{-1}w(t, z)w(t', -z)) = 1, \quad \forall t, t' \quad (8)$$

where we denote the symbol $\text{res}_z(\sum_i a_i z^i) = a_{-1}$. In fact, from the bilinear identity (8), solutions of the BKP hierarchy can be characterized by a single function $\tau(t)$ called τ -function such that[4]

$$\hat{w}(t, z) = \frac{\tau(t_1 - \frac{2}{z}, t_3 - \frac{2}{3z^3}, t_5 - \frac{2}{5z^5}, \dots)}{\tau(t)}. \quad (9)$$

From (7) and (9) the wave function $w(t, \lambda)$ can be expressed in terms of τ -function as

$$w(t, z) = \frac{X_B(t, z)\tau(t)}{\tau(t)},$$

where $X_B(t, z)$ is the so-called vertex operator, defined by [4]

$$X_B(t, z) = e^{\xi(t, z)}e^{-2D(t, z)} \equiv e^{\xi(t, z)}G(z),$$

with $D(t, z) = \sum_{n=0} z^{-2n-1}\partial_{2n+1}/(2n+1)$.

In the following we provide two useful identities associated with the tau function of the BKP hierarchy.

Proposition 1. [25] (*Fay identity*) *The tau function of the BKP hierarchy satisfies the Fay quadrisecant identity:*

$$\begin{aligned} & \sum_{(s_1, s_2, s_3)} \frac{(s_1 - s_0)(s_1 + s_2)(s_1 + s_3)}{(s_1 + s_0)(s_1 - s_2)(s_1 - s_3)} \tau(t + 2[s_2] + 2[s_3])\tau(t + 2[s_0] + 2[s_1]) \\ & + \frac{(s_0 - s_1)(s_0 - s_2)(s_0 - s_3)}{(s_0 + s_1)(s_0 + s_2)(s_0 + s_3)} \tau(t + 2[s_0] + 2[s_1] + 2[s_2] + 2[s_3])\tau(t) = 0 \end{aligned}$$

where (s_1, s_2, s_3) stands for cyclic permutations of s_1, s_2 and s_3 .

Proposition 2. [25] (*Differential Fay identity*) *The following equation holds.*

$$\begin{aligned} & \left(\frac{1}{s_2^2} - \frac{1}{s_1^2} \right) \{ \tau(t + 2[s_1] + 2[s_2])\tau(t + 2[s_2]) - \tau(t + 2[s_1] + 2[s_2])\tau(t) \} \\ &= \left(\frac{1}{s_2} + \frac{1}{s_1} \right) \{ \partial\tau(t + 2[s_2])\tau(t + 2[s_1]) - \partial\tau(t + 2[s_1])\tau(t + 2[s_2]) \} \\ &+ \left(\frac{1}{s_2} - \frac{1}{s_1} \right) \{ \tau(t + 2[s_1] + 2[s_2])\partial\tau(t) - \partial\tau(t + 2[s_1] + 2[s_2])\tau(t) \}. \end{aligned}$$

3 Additional symmetries and vertex operators

Based on the work of Orlov and Schulman [18], the Lax equation can be extended by introducing the Orlov-Schulman operator M defined by

$$M = W\Gamma W^{-1}, \quad \Gamma = \sum_{n=0} (2n+1)t_{2n+1}\partial^{2n},$$

which satisfies

$$\partial_{2n+1}M = [B_{2n+1}, M], \quad [L, M] = 1.$$

Thus the linear system (4) should be extended to

$$Lw = zw, \quad Mw = \partial_z w, \quad \partial_{2n+1}w = B_{2n+1}w.$$

Note that on the space of wave function $w(t, z)$, (L, M) is anti-isomorphic to (z, ∂_z) since $[z, \partial_z] = -1$. More general, one has

$$M^m L^l w = z^l \partial_z^m w, \quad L^l M^m w = \partial_z^m z^l w.$$

In fact, one can define the adjoint wave function $w^*(t, z) = (W^*)^{-1}e^{-\xi(t, z)} = -z^{-1}w_x(t, -z)$ and $M^* = (L^*)^{-1}\partial M\partial^{-1}L^*$ where we have use the fact that $\Gamma^* = \Gamma$ and (6). Then $[L^*, M^*] = [M, L]^* = -1$, and

$$L^*w^* = zw^*, \quad M^*w^* = -\partial_z w^*, \quad \partial_{2n+1}w^* = -B_{2n+1}^*w^*.$$

Motivated by the KP hierarchy, one can introduce a new set of parameters \hat{t}_{ml} so that additional symmetries of the BKP hierarchy can be expressed as

$$\hat{\partial}_{ml}W = -(A_{ml}(L, M))_-W, \tag{10}$$

where the generator $A_{ml}(L, M)$, due to (6), has the form [26, 25]

$$A_{ml}(L, M) = M^m L^l - (-1)^l L^{l-1} M^m L. \tag{11}$$

Let us introduce another generator $Y_B(\lambda, \mu)$ of additional symmetries as[26]

$$\begin{aligned} Y_B(\lambda, \mu) &= \sum_{m=0}^{\infty} \frac{(\mu - \lambda)^m}{m!} \sum_{l=-\infty}^{\infty} \lambda^{-l-m-1} (A_{m,m+l}(L, M))_-, \\ &= \sum_{m=0}^{\infty} \frac{(\mu - \lambda)^m}{m!} \sum_{l=-\infty}^{\infty} \lambda^{-l-m-1} (M^m L^{m+l} - (-1)^{m+l} L^{m+l-1} M^m L)_-. \end{aligned} \quad (12)$$

We mention that for $\lambda = \mu$, the generator $Y_B(\lambda, \lambda) = 2 \sum_{l=odd} \lambda^{-l-1} L_-^l$ corresponds to the resolvent operator of the BKP hierarchy.

On the other hand, recalling the vertex operator $X_B(\lambda, \mu)$ defined by

$$X_B(\lambda, \mu) = e^{-\xi(t, \lambda)} e^{\xi(t, \mu)} G(-\lambda) G(\mu)$$

which provides the infinitesimal Sato's Bäcklund transformations[4] on the space of tau function, namely, if $\tau(t)$ is a solution then $\tau(t) + \epsilon X_B(\lambda, \mu) \tau(t)$ is a solution as well. In fact, one can Taylor expand the vertex operator $X_B(\lambda, \mu)$ around $\mu = \lambda$ for large λ as

$$X_B(\lambda, \mu) = \sum_{m=0}^{\infty} \frac{(\mu - \lambda)^m}{m!} W^{(m)}(\lambda) = \sum_{m=0}^{\infty} \frac{(\mu - \lambda)^m}{m!} \sum_{l=-\infty}^{\infty} \lambda^{-m-l} W_l^{(m)},$$

where $W^{(m)}(\lambda) = \partial_\mu^m X_B(\lambda, \mu)|_{\mu=\lambda}$. Introducing the symbol $\alpha(z) = \sum_{n=odd} \alpha_n z^{-n}/n$ with

$$\alpha_n = \begin{cases} 2\partial/\partial t_n & n \in \mathbb{Z}_{odd}^+ \\ |n|t_{|n|} & n \in \mathbb{Z}_{odd}^- \end{cases}$$

where $\mathbb{Z}_{odd} = \mathbb{Z}_{odd}^+ \oplus \mathbb{Z}_{odd}^-$. Then α_n 's satisfy the commutation relations

$$[\alpha_n, \alpha_m] = 2n\delta_{n, -m}, \quad n \in \mathbb{Z}_{odd}.$$

The vertex operator $X_B(\lambda, \mu)$ can be expressed as

$$X_B(\lambda, \mu) =: e^{\alpha(\lambda) - \alpha(\mu)} :$$

where the normal ordering $::$ demands that $\alpha_{n>0}$ must be placed to the right of $\alpha_{n<0}$. Therefore,

$$W^{(m)}(\lambda) = \partial_\mu^m X_B(\lambda, \mu)|_{\mu=\lambda} =: \partial_\lambda^m e^{-\alpha(\lambda)} \cdot e^{\alpha(\lambda)} :=: F_m(-\partial_\lambda \alpha(\lambda)) :$$

where $F_m(u(z))$ is the so-called Faà di Bruno polynomials (see e.g. [7]) defined by the recurrence relations $F_{m+1}(u) = (\partial_z + u)F_m(u)$. For instance, $F_0 = 1, F_1 = u, F_2 = u' + u^2, F_3 = u'' + 3uu' + u^3$.

Lemma 3. [6] *The following formula*

$$W^{(m)}(\lambda) = \sum_{m_1+2m_2+\dots=m} \frac{m!}{m_1!m_2!\dots} : \left(-\frac{\partial_\lambda \alpha}{1!}\right)^{m_1} \left(-\frac{\partial_\lambda^2 \alpha}{2!}\right)^{m_2} \dots : \quad (13)$$

holds, where $m_i \geq 0$.

Proof. Introducing a generating function of F_m as $g(\lambda, z) = \sum_{m=0} F_m z^m / m!$, then

$$\begin{aligned}
g(\lambda, z) &= \sum_{m=0} \frac{(d_\lambda^m 1)}{m!} z^m, \quad d_\lambda = e^{\alpha(\lambda)} \cdot \partial_\lambda \cdot e^{-\alpha(\lambda)} \\
&= e^{\alpha(\lambda)} \cdot e^{z \partial_\lambda} e^{-\alpha(\lambda)} \\
&= e^{\alpha(\lambda)} e^{-\alpha(\lambda+z)} \\
&= \prod_{n=1} e^{-\partial_\lambda^n \alpha z^n / n!} \\
&= \sum_{m_1=0} \frac{\left(-\frac{\partial_\lambda \alpha}{1!}\right)^{m_1} z^{m_1}}{m_1!} \sum_{m_2=0} \frac{\left(-\frac{\partial_\lambda^2 \alpha}{2!}\right)^{m_2} z^{2m_2}}{m_2!} \dots \\
&= \sum_{m=0} \left(\sum_{m_1+2m_2+\dots=m} m! \frac{\left(-\frac{\partial_\lambda \alpha}{1!}\right)^{m_1}}{m_1!} \frac{\left(-\frac{\partial_\lambda^2 \alpha}{2!}\right)^{m_2}}{m_2!} \dots \right) \frac{z^m}{m!}.
\end{aligned}$$

□

The vertex operator generators $W_l^{(m)}$ can be easily computed as

$$\begin{aligned}
W_n^{(0)} &= \delta_{n,0}, \\
W_n^{(1)} &= \begin{cases} \alpha_n & n \in \mathbb{Z}_{\text{odd}} \\ 0 & n \in \mathbb{Z}_{\text{even}} \end{cases}, \\
W_n^{(2)} &= \begin{cases} -(n+1)\alpha_n & n \in \mathbb{Z}_{\text{odd}} \\ \sum_{i+j=n} : \alpha_i \alpha_j : & n \in \mathbb{Z}_{\text{even}} \end{cases}, \\
W_n^{(3)} &= \begin{cases} \sum_{i+j+k=n} : \alpha_i \alpha_j \alpha_k : + (n+1)(n+2)\alpha_n & n \in \mathbb{Z}_{\text{odd}} \\ -\frac{3}{2}(n+2) \sum_{i+j=n} : \alpha_i \alpha_j : & n \in \mathbb{Z}_{\text{even}} \end{cases}, \\
W_n^{(4)} &= \begin{cases} -2(n+3) \sum_{i+j+k=n} : \alpha_i \alpha_j \alpha_k : - (n+1)(n+2)(n+3)\alpha_n & n \in \mathbb{Z}_{\text{odd}} \\ \sum_{i+j+k+l=n} : \alpha_i \alpha_j \alpha_k \alpha_l : - \sum_{i+j=n} i j : \alpha_i \alpha_j : \\ -(2n^2 + 9n + 11) \sum_{i+j=n} : \alpha_i \alpha_j : & n \in \mathbb{Z}_{\text{even}} \end{cases},
\end{aligned}$$

etc.

A remarkable formula described below provides a bridge between additional symmetries acting on wave function and Sato's Bäcklund symmetries acting on τ function. This kind of formula was first derived for the KP hierarchy by Adler-Shiota-van Moerbeke[2, 3] and Dickey[6], and later for BKP by van de Leur[26] (see also [25]).

Theorem 4. [26, 25] *The following formula*

$$X_B(\lambda, \mu) w(t, z) = 2\lambda \left(\frac{\lambda - \mu}{\lambda + \mu} \right) Y_B(\lambda, \mu) w(t, z), \quad (14)$$

holds for the BKP hierarchy, where it should be understood that the vertex operator $X_B(\lambda, \mu)$ acting on $w(t, z)$ is generated by its action on the τ function.

We remark that the proof of (14) in [25] is based on a simple expression for the generator $Y_B(\lambda, \mu)$ and the differential Fay identity of the BKP hierarchy. Through the fermion-boson correspondence in the BKP hierarchy, a realization of Lie algebra $go(\infty)$ on $\mathbb{C}[t_1, t_3, t_5, \dots]$ is given by[4]

$$Z_B(\lambda, \mu) = \frac{1}{2} \frac{\mu + \lambda}{\mu - \lambda} (X_B(\lambda, \mu) - 1),$$

which after Taylor expanding around $\mu = \lambda$ for large λ has the form

$$Z_B(\lambda, \mu) = \sum_{m=0}^{\infty} \frac{(\mu - \lambda)^m}{m!} \sum_{l=-\infty}^{\infty} \lambda^{-m-l} Z_l^{(m+1)}.$$

It is easy to show that differential operators $Z_l^{(m)}$ are related to $W_l^{(m)}$ as

$$Z_l^{(1)} = W_l^{(1)}, \quad Z_l^{(m+1)} = \frac{W_l^{(m+1)}}{m+1} + \frac{1}{2} W_l^{(m)}, \quad m \geq 1,$$

and constitute an infinite-dimensional Lie algebra called $W_{1+\infty}^B$ -algebra which is a subalgebra of $W_{1+\infty}$ associated with the KP hierarchy.

4 p -reduced BKP and string equation

The so-called p -reduced BKP hierarchy[4] is defined by the Lax operator (2) such that

$$L^p = (L^p)_+, \quad p = \text{odd integer}. \quad (15)$$

This reduction must be compatible with the condition (3). For example, $(L^p)_{[0]} = 0$ for odd p and thus

$$(L^p)_+^* = (L^*)^p = (-1)^p \partial L^p \partial^{-1} = (-1)^p \partial (L^p)_+ \partial^{-1}$$

is a differential operator as well. Therefore, from the Lax equation (1), we have

$$\partial_{2n+1} L^p = \left[(L^p)_+^{\frac{2n+1}{p}}, L^p \right] \quad (16)$$

which implies that L^p is independent of the parameters t_{jp} for $j = 1, 3, 5, \dots$. For $p = 1$ case, we have $L = L_+ = \partial$ which is a trivial reduction ($u_i = 0$ for $\forall i$). The next case is $p = 3$, which contains the simplest nontrivial equation called Sawada-Kotera equation[20]:

$$u_t + 15(u^3 + uu_{xx})_x + u_{xxxxx} = 0$$

where $x = t_1$, $t = t_5$ and $u = 2(\log \tau)_{xx}$ with $\partial \tau / \partial t_{3j} = 0$ ($j = 1, 3, 5, \dots$).

In the following , we would like to characterize the solutions of the p -reduced BKP hierarchy constrained by the string equation

$$[L^p, P] = 1 \quad (17)$$

where P is a differential operator. From (11), we have

$$A_{1,1-k} = \begin{cases} kL^{-k} & k \in \mathbb{Z}_{\text{odd}} \\ 2ML^{1-k} - kL^{-k} & k \in \mathbb{Z}_{\text{even}} \end{cases} \quad (18)$$

Guided by the KP hierarchy[5], if we think the string equation (17) as the consequence of the addition flow equation $\hat{\partial}_{1,1-p}W = 0$ for odd p , then (10) and (18) show that $(A_{1,1-p})_- = pL^{-p} = 0$ which produces a contradictory result. In[27] van de Luer pointed out that one may consider the additional flow equation $\hat{\partial}_{1,1-2p}W = -(A_{1,1-2p})_-W = 0$ from which one gets

$$(ML^{1-2p})_- = pL^{-2p} \quad (19)$$

and hence the operator $Q \equiv (ML^{1-2p} - pL^{-2p})/2p$ is purely differential. Then

$$[L^{2p}, Q] = 1. \quad (20)$$

However, in view of the fact that $L^p = (L^p)_+$, we have

$$(ML^{1-p})_- = (ML^{1-2p}L^p)_- = pL^{-p}, \quad (21)$$

which provides the differential operator $P \equiv (ML^{1-p} - pL^{-p})/p$ for the string equation (17). Therefore, (19) can be regarded as the symmetry origin of the string equation (17), and thus we refer (19) to the pre-string equation. Taking the residue of (19) we obtain

$$\sum_{n=p} (2n+1)t_{2n+1} \text{res}(L^{2n-2p+1}) + (2p-1)t_{2p-1} = 0. \quad (22)$$

From the Sato's equation (5) and the formula (9), we have

$$\text{res}(L^{2n-2p+1}) = 2\partial_1 \partial_{2n-2p+1} \log \tau.$$

Substituting above back to (22) and integrating it over t_1 , yields

$$\left(\sum_{n=p} (2n+1)t_{2n+1} \frac{\partial}{\partial t_{2n+1-2p}} + \frac{(2p-1)}{2} t_1 t_{2p-1} + c \right) \tau(t) = 0. \quad (23)$$

In fact, from (19) and the p -reduced BKP hierarchy flows, we can prove a more general result of constraints on τ -function.

Proposition 5. *For the p -reduced Lax operator L^p constrained by the string equation, the following formulas hold for $m \geq 0$.*

$$(M^m L^{jp+m})_- = \begin{cases} \prod_{r=0}^{m-1} (p-r)L^{-2p} & j = -2 \\ \prod_{r=0}^{m-1} (p-r)L^{-p} & j = -1 \\ 0 & j = 0, 1, 2, \dots \end{cases} \quad (24)$$

$$(L^{jp+m-1} M^m L)_- = (-1)^m (M^m L^{jp+m})_- \quad (25)$$

with the proviso that the factor $\prod_{r=0}^{m-1} (p-r)$ should be set to 1 for $m = 0$.

Proof. We shall follow [1] to prove it by induction on m and j . For $m = 0$, the proof is obvious. For $m = 1, j = -2$, this is just the equation (19). Assume that the formula (24) holds up to some integer $m > 0$ for $j = -2$, then for $j \geq -1$, one sees that

$$\begin{aligned} (M^m L^{jp+m})_- &= ((M^m L^{m-2p})_- L^{(j+2)p})_- \\ &= \left(\prod_{r=0}^{m-1} (p-r)L^{-2p} L^{(j+2)p} \right)_- \\ &= \begin{cases} \prod_{r=0}^{m-1} (p-r)L^{-p} & j = -1 \\ 0 & j = 0, 1, 2, \dots \end{cases}. \end{aligned}$$

Next, for the $j = -2$, we have

$$\begin{aligned} (M^{m+1} L^{m+1-2p})_- &= (M^m M L^m L^{1-2p})_- \\ &= (M^m L^m M L^{1-2p})_- - m(M^m L^{m-2p})_- \\ &= (M^m L^m (M L^{1-2p})_-)_- - m(M^m L^{m-2p})_- \\ &= p(M^m L^{m-2p})_- - m(M^m L^{m-2p})_- \\ &= (p-m) \prod_{r=0}^{m-1} (p-r)L^{-2p} \\ &= \prod_{r=0}^m (p-r)L^{-2p} \end{aligned}$$

where the third equality is due to the fact that $M^m L^m$ is a differential operator. The formula (25) can be proved in a similar way. \square

We like to mention that a similar result as (24) for the CKP hierarchy has been derived in [12] to discuss the associated additional symmetries and string equation.

Proposition 6. *Let L^p be the Lax operator of the p -reduced BKP hierarchy constrained by the string equation (17). Then for $m \geq 0$ and $j \geq -2$,*

$$Z_{jp}^{(m+1)} \tau(t) = c \cdot \tau(t). \quad (26)$$

where c is a constant.

Proof. Using (12) and the Adler-Shiota-van Moerbeke formula (14), we have

$$\begin{aligned}
(A_{m,m+jp})_-\omega(t, z) &= \text{res}_\lambda (\lambda^{jp+m} \partial_\mu^m|_{\mu=\lambda} Y_B(\lambda, \mu) \omega(t, z)) \\
&= \text{res}_\lambda \left(\lambda^{jp+m-1} \partial_\mu^m|_{\mu=\lambda} \frac{1}{2} \frac{\lambda + \mu}{\lambda - \mu} X_B(\lambda, \mu) \omega(t, z) \right) \\
&= -\omega(t, z) (G(z) - 1) \text{res}_\lambda \left(\lambda^{jp+m-1} \partial_\mu^m|_{\mu=\lambda} \frac{Z_B(\lambda, \mu) \tau(t)}{\tau(t)} \right) \\
&= -\omega(t, z) (G(z) - 1) \left(\frac{Z_{jp}^{(m+1)} \tau(t)}{\tau(t)} \right).
\end{aligned}$$

On the other hand, from (24) and (25) we have

$$\begin{aligned}
(A_{m,m+jp})_-\omega(t, z) &= ((M^m L^{jp+m})_- - (-1)^{jp+m} (L^{jp+m-1} M^m L)_-) \omega(t, z) \\
&= (1 - (-1)^{jp}) (M^m L^{jp+m})_- \omega(t, z) \\
&= 2 \prod_{r=0}^{m-1} (p-r) L^{-p} \omega(t, z) \delta_{j,-1} \\
&= 2 \prod_{r=0}^{m-1} (p-r) z^{-p} \omega(t, z) \delta_{j,-1}.
\end{aligned}$$

Noticing that

$$2z^{-p} = -(p(t_p - 2z^{-p}/p) - pt_p) = -(G(z) - 1) \frac{Z_{-p}^{(1)} \tau(t)}{\tau(t)},$$

and hence

$$(G(z) - 1) \left(\frac{Z_{jp}^{(m+1)} \tau(t)}{\tau(t)} - \prod_{r=0}^{m-1} (p-r) \delta_{j,-1} \frac{Z_{-p}^{(1)} \tau(t)}{\tau(t)} \right) = 0.$$

Since $(G(z) - 1)f(t) = 0$ implies $f(t)$ is a constant, we have

$$\left(Z_{jp}^{(m+1)} - \prod_{r=0}^{m-1} (p-r) \delta_{j,-1} Z_{-p}^{(1)} \right) \tau(t) = c \cdot \tau(t).$$

Now we can drop the term involving $Z_{-p}^{(1)} = pt_p$ without harm because the p -reduced BKP hierarchy does not depend on the variables t_p . \square

5 String equation as the lowest Virasoro constraint

In this section, we like to discuss the algebraic structure of the equation (26). Given a infinite-dimensional algebra $W_{1+\infty}^B$ defined by the vertex operator, one can introduce two subalgebras as follows:

$$W_p^B = \{\text{generated by } W_{jp}^{(m)}, 1 \leq m \leq p, j \in \mathbb{Z}, t_p = t_{3p} = \cdots = 0\}$$

and its truncated sub-algebra:

$$W_p^{B+} = \{\text{generated by } W_{jp}^{(m)}, 1 \leq m \leq p, j \geq -2, t_p = t_{3p} = \dots = 0\}$$

We shall show that the τ function satisfying the p -reduced BKP hierarchy and the string equation is a null-vector of the W_p^{B+} -algebra. Since the algebra W_p^+ for the KP hierarchy has no central term[10], thus we expect that the subalgebra $W_p^{B+} \subset W_p^+$ has also no central term. To see this, we have to properly combine the generators $Z_{jp}^{(m)}$ so that every redefined element $\mathbf{W}_n^{(m)}$ of W_p^{B+} can be expressed as a commutator of two elements of W_p^{B+} . As a consequence, the constant c in (26) can be removed and

$$\mathbf{W}_n^{(m)} \tau(t) = 0, \quad 1 \leq m \leq p, n \geq -m + 1. \quad (27)$$

We remark that the condition for the subscript n in (27) is due to the fact that for higher-spin generators $\mathbf{W}_n^{(m)}$, one has

$$[\mathbf{W}_{-1}^{(2)}, \mathbf{W}_n^{(m)}] = (-(m-1) - n) \mathbf{W}_{n-1}^{(m)},$$

and thus, under bracketing with generators $\mathbf{W}_{-1}^{(2)}$, one can reach the lowest one $\mathbf{W}_{-m+1}^{(m)}$ which commutes with $\mathbf{W}_{-1}^{(2)}$. Let us demonstrate the first few W -constraints.

For $m = 0$, (26) shows that

$$Z_{jp}^{(1)} \tau(t) = W_{jp}^{(1)} \tau(t) = 2\partial\tau(t)/\partial t_{jp} = 0, \quad j = 1, 3, 5, \dots$$

which is just the condition $L^p = (L^p)_+$ for p -reduced BKP hierarchy. Thus we have

$$\mathbf{W}_n^{(1)} \tau(t) = W_{(2n+1)p}^{(1)} \tau(t) = 0, \quad n \geq 0$$

For $m = 1$, (26) shows that $Z_{2kp}^{(2)} \tau = c_k^{(2)}(p) \tau$ with $k \geq -1$ where

$$Z_{2kp}^{(2)} = 2 \sum_{n=0}^{kp-1} \partial_{2n+1} \partial_{2kp-2n-1} + 2 \sum_{n=0}^{\infty} (2n+1) t_{2n+1} \partial_{2kp+2n+1}$$

for $k \geq 0$ and

$$\begin{aligned} Z_{2kp}^{(2)} &= \frac{1}{2} \sum_{n=0}^{-kp-1} (2n+1)(-2kp-2n-1) t_{2n+1} t_{-2kp-2n-1} \\ &\quad + 2 \sum_{n=0}^{\infty} (2n+1-2kp) t_{2n+1-2kp} \partial_{2n+1} \end{aligned}$$

for $k < 0$. Define $l_k = Z_{2kp}^{(2)}/4p$ with $k \geq -1$ then l_k satisfy centerless Virasoro algebra $[l_n, l_m] = (n-m)l_{n+m}$ except that

$$[l_1, l_{-1}] = 2 \left(l_0 + \frac{1}{16p} + \frac{p^2-1}{24p} \right).$$

This means that the constants $c_{k \neq 0}^{(2)}(p) = 0$ and $c_0^{(2)}(p) = -(1/4 + (p^2 - 1)/6)$. If we redefine a new set of operators as

$$\mathbf{W}_n^{(2)} \equiv \mathbf{L}_n = l_n + \delta_{n,0} \left(\frac{1}{16p} + \frac{p^2 - 1}{24p} \right), \quad n \geq -1$$

then we have $[\mathbf{L}_n, \mathbf{L}_m] = (n - m)\mathbf{L}_{n+m}$ for $m, n \geq -1$. Therefore, the Virasoro constraint becomes

$$\mathbf{W}_n^{(2)} \tau(t) = 0, \quad n \geq -1.$$

In particular, the lowest Virasoro constraint $\mathbf{L}_{-1} \tau = 0$ is given by

$$\left(\frac{1}{2} \sum_{n=0}^{p-1} (2n+1)(2p-2n-1)t_{2n+1}t_{2p-2n-1} + 2 \sum_{n=0} (2n+1+2p)t_{2n+1+2p}\partial_{2n+1} \right) \tau = 0$$

which is just the pre-string equation (23). Moreover, when n extends to all integers, $\mathbf{W}_n^{(2)}$ indeed constitute the generators of a standard Virasoro algebra with central charge $c = p$, namely,

$$[\mathbf{L}_n, \mathbf{L}_m] = (n - m)\mathbf{L}_{n+m} + \delta_{n+m,0} \frac{n^3 - n}{12} p.$$

For $m = 2$, we may define the spin-3 generators as $\mathbf{W}_n^{(3)} = Z_{(2n+1)p}^{(3)}$ which indeed satisfy the commutation relation

$$[\mathbf{L}_n, \mathbf{W}_m^{(3)}] = (2n - m)\mathbf{W}_{n+m}^{(3)}$$

and the constraint equation

$$\mathbf{W}_n^{(3)} \tau(t) = 0, \quad n \geq -2.$$

The higher-spin generators can be treated in a similar manner.

6 Grassmannian description of the string equation

In this section we like to give a geometric description of the string equation for the p -reduced BKP hierarchy. Let H be a Hilbert space defined by formal power series in z that can be decomposed into two infinite-dimensional subspaces as $H = H_+ \oplus H_-$ where

$$H_+ = \text{span}\{z^0, z^1, z^2, \dots\}, \quad H_- = \text{span}\{z^{-1}, z^{-2}, z^{-3}, \dots\}.$$

The Grassmannian Gr is defined by the set of all subspaces $V \subset H$ with the following conditions:[19, 22]

$$Gr = \{V | V \subset H, p_+|_V : V \rightarrow H_+ \text{ (Fredholm)}, p_-|_V : V \rightarrow H_- \text{ (compact)}\}$$

where p_{\pm} are projection operators. If $p_+|_V : V \rightarrow H_+$ is a bijection, then V is called transversal to H_- , or transversal for short (i.e. V belongs to the big cell $Gr_0 \subset Gr$). The KP hierarchy can be regarded as a simple dynamical system on Gr (see [19] for the detail).

The BKP hierarchy is defined by the subvariety $Gr^B \subset Gr$ (see e.g. [23]) so that an infinite-dimensional plane $V^0 \in Gr^B$ can be represented as follows [19, 22]:

$$\begin{aligned} V^0 &= \text{span}\{\omega(t, z)|_{t=0}, \partial_x \omega(t, z)|_{t=0}, \partial_x^2 \omega(t, z)|_{t=0}, \dots\} \\ &= \text{span}\{\omega(t, z), \text{for all } t \in \mathbb{C}^\infty\}. \end{aligned}$$

where $t = (t_1, t_3, t_5, \dots)$ and $\omega(t, z)$ satisfies the bilinear equation (8). For the p -reduced BKP hierarchy, $\partial_p W = -(L^p)_- W$ and the wave function $\omega(t, z)$ associated with V^0 satisfies

$$\partial_p \omega(t, z) = (L^p)_+ \omega(t, z) = L^p \omega(t, z) = z^p \omega(t, z) \in V^0$$

Therefore

$$z^p V^0 \subset V^0.$$

On the other hand, the string equation (17) can be traced back to the Sato equation of the additional flow $\hat{\partial}_{1,1-2p} W = -(A_{1,1-2p})_- W = 0$ which follows

$$\hat{\partial}_{1,1-2p} \omega(t, z) = -(A_{1,1-2p})_- \omega(t, z) = 0$$

and hence

$$\begin{aligned} Q\omega(t, z) &= \frac{1}{2p}(ML^{1-2p} - pL^{-2p})\omega(t, z) \\ &= \frac{1}{2p}(z^{1-2p}\partial_z - pz^{-2p})\omega(t, z) \\ &\equiv A_{2p}\omega(t, z) \in V^0. \end{aligned}$$

Therefore, the plane $V^0 \in Gr^B$ associated with $\omega(t, z)$ of p -reduced BKP hierarchy constrained by the string equation $[L^p, P] = 1$ is invariant under the action of differential operators L^p and $Q \equiv \frac{1}{2p}(ML^{1-2p} - pL^{-2p})$. They act as z -operators:

$$L^p \mapsto z^p, Q \mapsto A_{2p} \equiv z^p \frac{d}{dz^{2p}} z^{-p}$$

and

$$z^p V^0 \subset V^0, A_{2p} V^0 \subset V^0, \text{ with } [A_{2p}, z^{2p}] = 1. \quad (28)$$

The above discussions enable us to transform the original problem for solving the solution of the p -reduced BKP hierarchy constrained by the string equation to that described in z -operators in the context of Grassmannian.

7 Concluding Remarks

We have investigated the string equation of the BKP hierarchy from additional symmetries point of view. We show that the p -reduced BKP hierarchy constrained by the string equation can be formulated in terms of Lax and Orlov-Schulman operators as what has been done for the KP hierarchy. In particular, the invariance of the additional symmetry with respect to the $\hat{t}_{1,1-2p}$ -flow is crucial for obtaining the string equation which together with the Adler-Shiota-van Moerbeke formula implies the W -constraints of the τ function. Furthermore, we show that the Lax-Orlov-Schulman formulation of the p -reduced BKP hierarchy with string equation can be transformed to that of Sato Grassmannian in terms of the spectral parameter z . In view of the works by Kac and Schwarz[14, 21], the Grassmannian description might provide a starting point for investigating the existence problem of BKP solutions characterized by the string equation. We hope to address this issue in our future work.

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